Lecture 4: Mean-Variance Theory and CAPM

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Mean-Variance Theory (MVT)

- Economist Harry Markowitz introduced MVT in a 1952, for which he was awarded a Nobel Prize in 1990
- The assumption made by the theory is that only the mean and variance of the investment return matter
- The MVT is a method of constructing a portfolio that generates a maximum return for a given level of risk or a minimum risk for a stated return
- Although MVT has some limitations, it continues to be a cornerstone for portfolio managers
- **Remark:** Markets do not have to be complete

Reduction of Portfolio Risk Through Diversification

- The correlation between securities impacts on the overall variance of the portfolio
- In general, higher correlation between assets in a portfolio will result in a higher portfolio variance. The inverse is also true.

$$\sigma^2_{R_p} = \sum_{i=1}^{n} (w_i)^2 \sigma^2_{R_i} + \sum_{i=1}^{n} \sum_{k \neq i} w_i w_k cov(R_i, R_k)$$

- If the weights are positive, a lower covariance between the securities implies lower portfolio variance.
- By combining various risky assets we can improve the portfolio's return-risk characteristics, resulting in a better trade-off.

Reduction of Portfolio Risk Through Diversification

- If the returns are uncorrelated, i.e. with covariance zero, the expression above reduces to:

$$\sigma^2_{R_p} = \sum_{i=1}^{n} (w_i)^2 \, \sigma^2_{R_i}$$

- To understand this more clearly, let us assume that we invest equal amounts in each asset

$$w_i = \frac{1}{n}$$

and $\sigma^2_{R_i}$ is the same for all assets, i.e. $\sigma^2_{R_i} = \sigma^2_R$

Reduction of Portfolio Risk Through Diversification

- Thus
$$\sigma^2_{R_p} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2_R = \frac{1}{n}\sigma^2_R$$
- As *n* gets larger, the variance tends to zero.
- In other words, the more independent assets in the portfolio, the lesser the variance and hence the risk.
- That is to say; risk can be minimized by diversification.

Calculating the Optimal Portfolio

- Consider a portfolio made up of assets A and B. Let σ_{R_A} and σ_{R_B} represent the standard deviation of the assets rate of return, respectively.
- Also, let w_A and let w_B represent the proportion of funds invested in each asset.
- We describe below how we can make the portfolio of the two assets optimal.
- An optimal portfolio minimizes risk for a given level of return.
- The optimal portfolio is created by investing the right proportion of funds in the respective assets making up the portfolio.
- To calculate the optimal portfolio, we, therefore, need to compute the appropriate asset allocations that ensure minimum risk.

Calculating the Optimal Portfolio

- The portfolio return is $R_P = w_A R_A + w_B R_B$ where the weights sum to one, $w_A + w_B = 1$
- To determine the asset allocation that minimizes portfolio risk, we differentiate the portfolio variance

$$\sigma^2_{R_P} = w_A^2 \sigma^2_{R_A} + (1 - w_A)^2 \sigma^2_{R_B} + 2 w_A (1 - w_A)\, cov(R_A, R_B)$$

  w.r.t. w_A
- Equate the first derivative to zero and solve for the minimum point
- The value w_A that minimizes portfolio risk, is

$$w_A = \frac{\sigma^2_{R_B} - cov(R_A, R_B)}{\sigma^2_{R_A} + \sigma^2_{R_B} - 2cov(R_A, R_B)}$$

Case 1: Without a risk free asset

- A portfolio is a **mean variance frontier portfolio** if it has the minimum variance among the portfolios that have the same expected rate of return.
- Let the vector of asset returns be $\mathbf{R}$.
- Let $\mathbf{E}$ be the vector of mean returns, $\mathbf{E} \equiv E(\mathbf{R})$, and
- Let Σ the variance-covariance matrix $\Sigma = E\left[(\mathbf{R}-\mathbf{E})(\mathbf{R}-\mathbf{E})^T\right]$.
- A portfolio is defined by its weights $\mathbf{w}$ on the securities.
- The portfolio return is $\mathbf{w}^T\mathbf{R}$ where the weights sum to one, $\mathbf{w}^T\mathbf{1} = \mathbf{1}$.

Case 1: Without a risk free asset

- The problem "**choose a portfolio to minimize variance for a given mean**" is then

$$\min_{\{\mathbf{w}\}} \left\{ \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w} \right\} \text{ s.t. } \mathbf{w}^T \mathbf{E} = \mu \text{ and } \mathbf{w}^T \mathbf{1} = 1.$$

- The objective function is convex in **w** and the restrictions are linear in **w**.
- Let the Lagrange multipliers on the constraints be λ and δ.

Case 1: Without a risk free asset

- Lagrangian

$$L = \frac{1}{2}\mathbf{w}^T\Sigma\mathbf{w} + \lambda\left(\mu - \mathbf{w}^T\mathbf{E}\right) + \delta\left(1 - \mathbf{w}^T\mathbf{1}\right)$$

- The first order conditions are then

$$\Sigma\mathbf{w} - \lambda\mathbf{E} - \delta\mathbf{1} = 0$$

$$\Rightarrow \mathbf{w} = \Sigma^{-1}(\lambda\mathbf{E} + \delta\mathbf{1}).$$

- Replacing in the constraints,

$$\mathbf{E}^T\mathbf{w} = \mathbf{E}^T\Sigma^{-1}(\lambda\mathbf{E} + \delta\mathbf{1}) = \mu$$

$$\mathbf{1}^T\mathbf{w} = \mathbf{1}^T\Sigma^{-1}(\lambda\mathbf{E} + \delta\mathbf{1}) = 1$$

or

$$\begin{bmatrix} \mathbf{E}^T\Sigma^{-1}\mathbf{E} & \mathbf{E}^T\Sigma^{-1}\mathbf{1} \\ \mathbf{1}^T\Sigma^{-1}\mathbf{E} & \mathbf{1}^T\Sigma^{-1}\mathbf{1} \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

where $A = \mathbf{E}^T\Sigma^{-1}\mathbf{E}$; $B = \mathbf{E}^T\Sigma^{-1}\mathbf{1}$; $C = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$

$$\begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

Case 1: Without a risk free asset

Remember: For any nonsingular matrix $\mathbf{A}$:
$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj\left(\mathbf{A}\right)$, where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$ and $adj\left(\mathbf{A}\right)$ is defined as the transpose of the matrix $[A_{ij}]$, where A_{ij} is the cofactor of the element a_{ij}.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1} = \frac{1}{AC - B^2} \begin{bmatrix} C & -B \\ -B & A \end{bmatrix}$$

$$\lambda = \frac{C\mu - B}{AC - B^2}$$

$$\delta = \frac{A - B\mu}{AC - B^2}$$

Case 1: Without a risk free asset

- Plugging in

$$\mathbf{w} = \Sigma^{-1}(\lambda \mathbf{E} + \delta \mathbf{1})$$

we get the portfolio weights

$$\mathbf{w} = \Sigma^{-1}\left(\frac{C\mu - B}{AC - B^2}\mathbf{E} + \frac{A - B\mu}{AC - B^2}\mathbf{1}\right)$$

$$\mathbf{w} = \Sigma^{-1}\frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

to get the portfolio variance we compute

$$\Sigma \mathbf{w} = \frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

Case 1: Without a risk free asset

- thus the portfolio variance is

$$\mathbf{w}^T \Sigma \mathbf{w} = \frac{\mathbf{w}^T \mathbf{E}(C\mu - B) + \mathbf{w}^T \mathbf{1}(A - B\mu)}{AC - B^2}$$

or

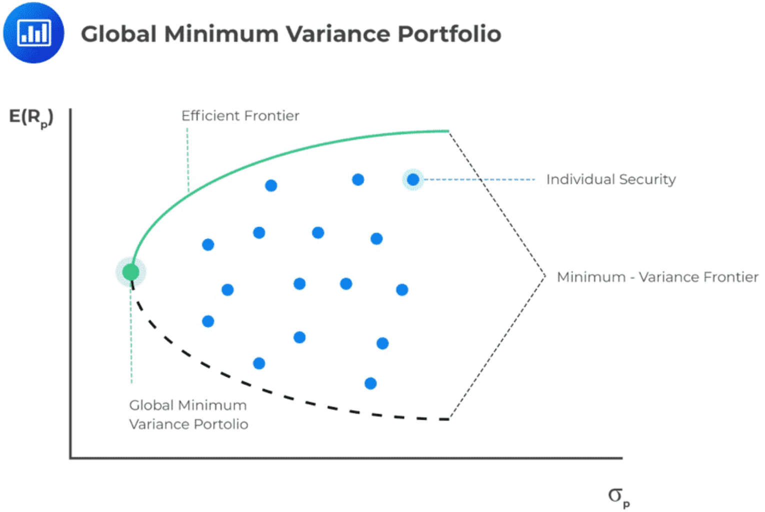
$$var(R_p) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}$$

since $\mathbf{w}^T \mathbf{E} = \mu$ and $\mathbf{w}^T \mathbf{1} = 1$.

- The variance is a quadratic function of the mean
- The variance is a parabola
- The standard deviation (volatility) is a hyperbola

Remark: The square root of a parabola is a hyperbola

Case 1: Without a risk free asset



Case 1: Without a risk free asset

- Example: Lets take

$$\Sigma = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 1.2 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}$$

$$A = \mathbf{E}^T \Sigma^{-1} \mathbf{E} = 1.7934$$

$$B = \mathbf{E}^T \Sigma^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.6264$$

$$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.4945$$

Case 1: Without a risk free asset

Let

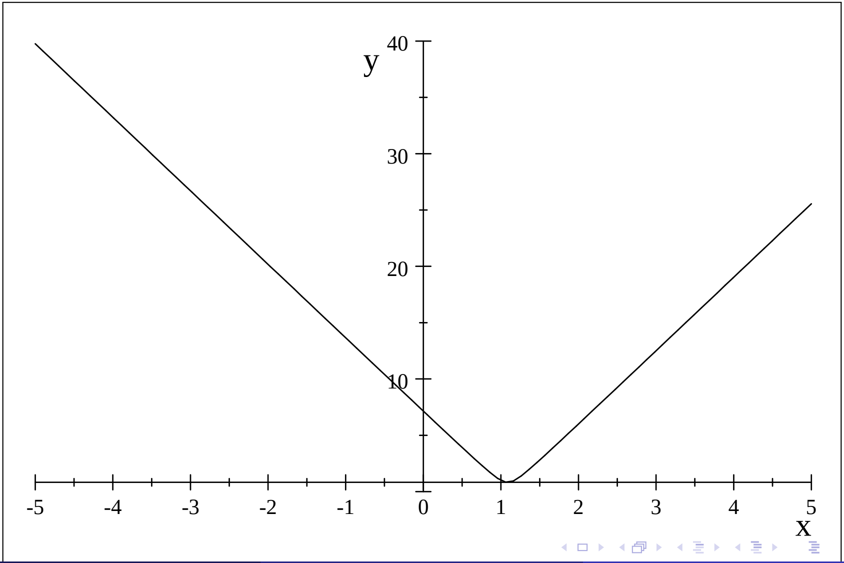
$$y^2 = var(R_p) \text{ and } x = \mu$$

$$y^2 = \frac{Cx^2 - 2Bx + A}{AC - B^2}$$

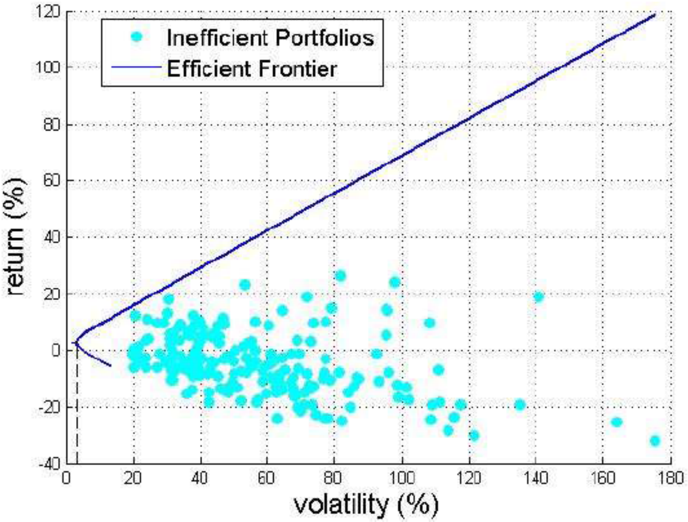
$$y^2 = \frac{1.4945x^2 - 2(1.6264)x + 1.7934}{1.7934(1.4945) - (1.6264)^2}$$

$$\begin{aligned} y &= \left(\frac{1.4945x^2 - 2(1.6264)x + 1.7934}{1.7934(1.4945) - (1.6264)^2}\right)^{1/2} \\ &= \sqrt{42.628x^2 - 92.780x + 51.153} \end{aligned}$$

Minimum variance portfolio (inverted)



Case 1: Without a risk free asset



- Example: If the mean is one: $x = 1$

The standard deviation is $y = \sqrt{42.628 - 92.780 + 51.153} = 1.0005$

- **minimum-variance portfolio**

solves the problem

$$\min_{\{\mu\}} \{var(R_p)\} = \frac{C\mu^2 - 2B\mu + A}{AC - B^2}$$

FOC

$$\frac{2C\mu - 2B}{AC - B^2} = 0$$

$$\mu^{\text{min var}} = B/C$$

Case 1: Without a risk free asset

As

$$\mathbf{w} = \Sigma^{-1}\frac{\mathbf{E}(C\mu - B) + \mathbf{1}(A - B\mu)}{AC - B^2}$$

The weights of the minimum variance portfolio are

$$\mathbf{w} = \Sigma^{-1}\frac{\mathbf{E}(C\frac{B}{C} - B) + \mathbf{1}(A - B\frac{B}{C})}{AC - B^2}$$

$$\mathbf{w} = \Sigma^{-1}\frac{\mathbf{1}\frac{1}{C}(AC - B^2)}{AC - B^2}$$

$$\mathbf{w} = \Sigma^{-1}\mathbf{1}/(\mathbf{1}^T\Sigma^{-1}\mathbf{1})$$

Case 1: Without a risk free asset

- When does a mean-variance frontier exist? In our proof the only assumption made was that Σ^{-1} exists.
- Theorem: So long as the variance-covariance matrix of returns is non-singular, there is a mean-variance frontier.
- This rules out two returns perfectly correlated but yielding different means

Case 1: Without a risk free asset

- *Result:* The frontier is spanned by any two frontier returns.
- Since $\mathbf{w}$ is a linear function of μ,

$$\mathbf{w} = \Sigma^{-1} \frac{\mathbf{E}(\mathbf{C}\mu - \mathbf{B}) + \mathbf{1}(\mathbf{A} - \mathbf{B}\mu)}{\mathbf{AC} - \mathbf{B}^2}$$

- A portfolio on the frontier with mean $\mu_3 = \lambda\mu_1 + (1-\lambda)\mu_2$ can be achieved with a portfolio of weights λ and $(1-\lambda)$ on two distinct portfolios on the frontier with two distinct mean returns μ_1 and μ_2.
- The weights on a third portfolio are given by $\mathbf{w}_3 = \lambda\mathbf{w}_1 + (1-\lambda)\mathbf{w}_2$.

Case 2: With a risk free asset

- assume now that there is a risk free asset with return R_f
- let $\mathbf{w} = (w_1, w_2, ..., w_n)$ be the vector of weights on the n risky assets and so that $1 - \sum_{i=1}^{n} w_i$ is the weight on the risk free security
- the investor's optimization problem is

$$\min_{\{\mathbf{w}\}} \left\{ \frac{1}{2} \mathbf{w}^T \Sigma \mathbf{w} \right\} \text{ s.t. } \left(1 - \sum_{i=1}^{n} w_i \right) R_f + \mathbf{w}^T \mathbf{E} = \mu$$

the optimal $\mathbf{w}$ is

$$\mathbf{w} = \zeta \Sigma^{-1} \left(\mathbf{E} - R_f \mathbf{1} \right)$$

where

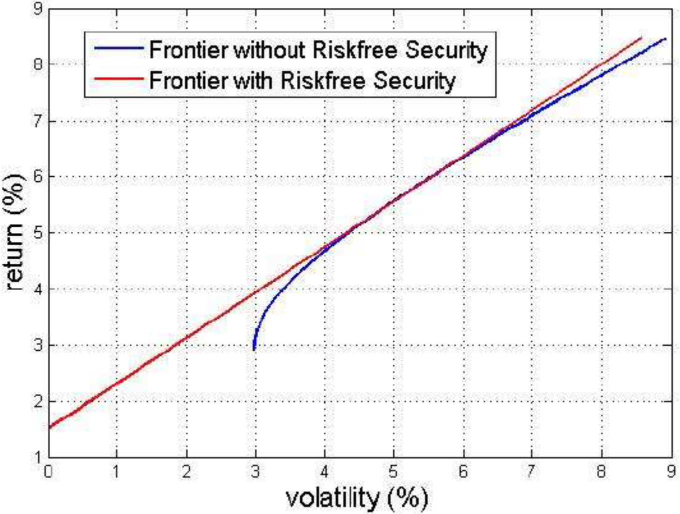
$$\xi = \sigma^2_{\min} / (\mu - R_f)$$

and $\sigma^2_{\min}$ is the minimized variance (twice the value of the minimized objective function)

$$\sigma^2_{\min} = \frac{(\mu - R_f)^2}{(\mathbf{E} - R_f \mathbf{1})^T \Sigma^{-1} (\mathbf{E} - R_f \mathbf{1})}$$

- It follows that $\sigma_{\min}$ is a linear function of μ.

Case 2: With a risk free asset



Case 2: With a risk free asset

- The risk-free rate is either below or at the point of minimum variance on the risky frontier. Why can't it be above? Explain.
- Suppose now that $R_f < \overline{R}_{mv}$
- The efficient frontier becomes a straight line that is tangential to the risky efficient frontier and with the y-intercept equal to R_f
- This result says that every investor optimally choose to invest in a combination of risk-free security and a single risky portfolio, i.e. the tangency portfolio

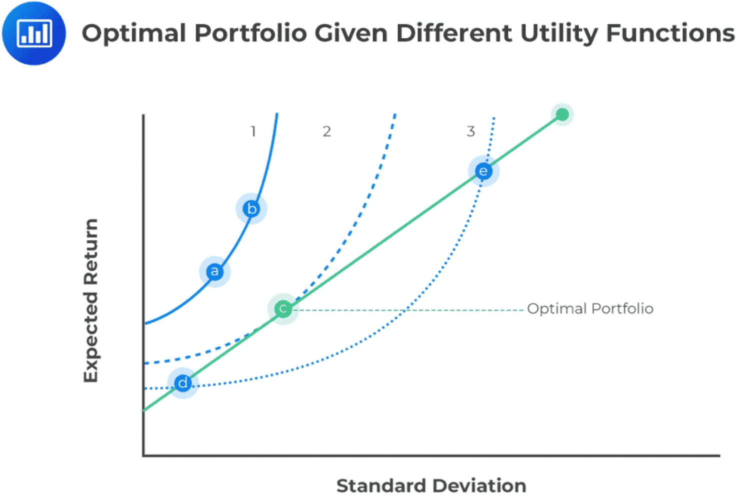
Case 2: With a risk free asset

- Exercise: Show without using the formula for $\sigma^2_{\min}$ that the efficient frontier is indeed a straight line.
- Hint: consider forming a portfolio of the risk-free with any risky portfolio. Show that the mean and standard deviation of the portfolio varies linearly with a share α, where α is the weight on the risky security.
- Exercise: Describe the efficient frontier if no borrowing is allowed

Optimal Portfolio

- Assume the investor is a mean-variance optimizer
- The investor's indifference curves have a negative slope on the space (μ, σ) because investors are willing to hold a portfolio with more risk only if the portfolio expected return is higher
- The investor's indifference curves are convex because in order to accept equal increases in risk, the investor requires larger increases in the expected return the higher the risk of his portfolio
- By imposing the investor's indifference curves (with negative slope and convex) on the space (μ, σ) get the optimal portfolio

Optimal Portfolio



CAPM

- Each investor will hold the same tangency portfolio of risky securities in conjunction with a position in the risk-free asset
- Because the tangency portfolio is held by all investors and because markets must clear, we can identify this portfolio as the market portfolio
- The **efficient frontier** is termed the **capital market line**

CAPM

- Let R_m and $E[R_m]$ denote the return and expected return, respectively, of the market, i.e. tangency, portfolio
- The central insight of the CAPM is that in equilibrium the **riskiness of an asset** is not measured by the standard deviation of its return but by its **beta**.
- In particular, there is a linear relationship between the expected return, $E[R]$ say, of any security (or portfolio) and the expected return of the market portfolio. It is given by

$$E[R] = R_f + \beta\left[E\left[R_m\right] - R_f\right]$$

where $\beta = Cov(R, R_m)/Var(R_m)$.

CAPM

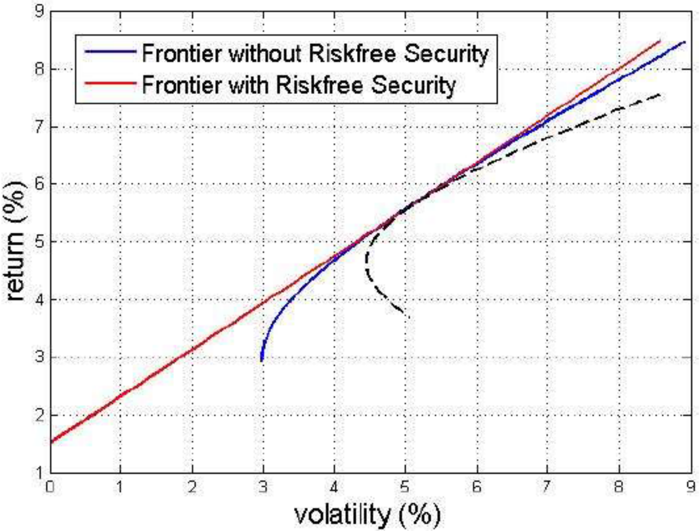
- In order to prove this consider a portfolio with weights α and weight $1-\alpha$ on the risky security and market portfolio, respectively.
- Let R_α denote the (random) return of this portfolio as a function of α

$$E[R_\alpha] = \alpha E\left[R\right] + (1-\alpha)E\left[R_m\right]$$

$$\sigma^2_{R_\alpha} = \alpha^2\sigma^2_R + (1-\alpha)^2\sigma^2_{R_m} + 2\alpha(1-\alpha)cov(R, R_m)$$

- As α varies, the mean and standard deviation, $(E[R_\alpha], \sigma^2_{R_\alpha})$ trace out a curve that cannot (why?) cross the efficient frontier. This curve is depicted as the dashed curve below.

CAPM


Frontier without Riskfree Security
Frontier with Riskfree Security
return (%)
volatility (%)
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9

CAPM

- At $\alpha = 0$ this curve must be tangent to the capital market line.

Therefore the slope of the curve at $\alpha = 0$ must equal the slope of the capital market line. Using the equations for $E[R_\alpha]$ and $\sigma^2_{R_\alpha}$ we see the slope is given by

$$\frac{dE[R_\alpha]}{d\sigma_{R_\alpha}} = \frac{\frac{dE[R_\alpha]}{d\alpha}}{\frac{d\sigma_{R_\alpha}}{d\alpha}}$$

where

$$\frac{dE[R_\alpha]}{d\alpha} = E[R] - E[R_m]$$

$$\frac{d\sigma_{R_\alpha}}{d\alpha} = \frac{1}{2}\left(\sigma^2_{R_\alpha}\right)^{-\frac{1}{2}}\left[2\alpha\sigma^2_R - 2(1-\alpha)\sigma^2_{R_m} + (2-4\alpha)cov(R, R_m)\right]$$

CAPM

(evaluate the derivatives when $\alpha = 0$)

$$\frac{dE[R_\alpha]}{d\alpha} = E[R] - E[R_m]$$

$$\frac{d\sigma_{R_\alpha}}{d\alpha} = (\sigma_{R_m})^{-1}\left[-\sigma_{R_m}^2 + cov(R, R_m)\right]$$

$$\frac{dE[R_\alpha]}{d\sigma_{R_\alpha}} = \frac{\frac{dE[R_\alpha]}{d\alpha}}{\frac{d\sigma_{R_\alpha}}{d\alpha}} = \frac{\sigma_{R_m}(E[R] - E[R_m])}{-\sigma_{R_m}^2 + cov(R, R_m)}$$

The slope of the capital market line is $(E[R_m] - R_f)/\sigma_{R_m}$
Equating the two

$$\frac{\sigma_{R_m}(E[R] - E[R_m])}{-\sigma_{R_m}^2 + cov(R, R_m)} = \frac{E[R_m] - R_f}{\sigma_{R_m}}$$

gives

$$E[R] = R_f + \beta[E[R_m] - R_f]$$

CAPM

- The CAPM result is one of the most famous results in all of finance and, even though it arises from a simple one-period model, it provides considerable insight to the problem of asset-pricing.
- For example, it is well-known that riskier securities should have higher expected returns in order to compensate investors for holding them.
- But how do we measure risk? Counter to the prevailing wisdom at the time the CAPM was developed, the riskiness of a security is not measured by its return volatility. Instead it is measured by its beta, which is proportional to its covariance with the market portfolio.
- Exercise: Why does the CAPM result not contradict the mean-variance problem formulation where investors do measure a portfolio's risk by its variance?

CAPM

- The CAPM is an example of a so-called 1-factor model with the market return playing the role of the single factor.
- Other factor models can have more than one factor. For example, the Fama-French model has three factors, one of which is the market return.
- Many empirical research papers have been written to test the CAPM.

CAPM

- Such papers usually perform regressions of the form

$$R_i - R_f = \alpha_i + \beta_i (R_m - R_f) + \epsilon_i$$

where ϵ_i is the idiosyncratic or residual risk which is assumed to be independent of R_m

- If the CAPM holds then we should be able to reject the hypothesis that $\alpha_i \neq 0$
- The evidence in favor of the CAPM is mixed. But the language inspired by the CAPM is now found throughout finance
- For example, we use β's to denote factor loadings and α's to denote excess returns even in non-CAPM settings